Logical Model of Graded Beliefs for a Persuasion Theory

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Abstract. The aim of the article is to introduce the model of beliefs' gradation for a persuasion theory and study the possibilities of its formalization. Thus, we formulate the requirements which the adequate description of graded beliefs should fulfil as well as show their significance for representing the process of convincing. Then, we provide a comprehensive survey of methods currently widely used to model the degrees of beliefs as well as evaluate them from the point of view of the postulated requirements.

1 Introduction

The paper reports on our research conducted in the field of gradation of beliefs. We understand this notion as a degree of uncertainty with respect to the truthfulness of a given formula. The topic discussed in the article is a part of the more general project in which we aim to built the **formal theory of persuasion**.[3] The motivation for our studies comes from computer science - we want to simulate the process of (successful) convincing. We define persuasion as an action which is initiated by a conflict of opinion and aimed to influence beliefs of others. Using philosophical reflection and different logical tools we consecutively investigate various aspects of this process including the degrees of uncertainty. The contribution of this paper is that we formulate the requirements which in our opinion should be satisfied by a model of graded beliefs in order to apply it to a persuasion theory as well as we rigorously research and evaluate selected formalisms under these requirements.

We need to understand the importance of the role that graded beliefs play in persuasion as well as be aware of the possibilities and difficulties of their formalization. Thus, we should take two issues into consideration here. First, where do we find beliefs gradation in the process of convincing? Since the persuader aims to influence someone's beliefs she does her best trying different tactics: she expresses verbal arguments or performs nonverbal arguments (hits, smiles, cries, etc.). However, in real-life practice it is not always the case that these actions results with a full success. Instead, the persuader often succeeds even though she partly convinces others. This means that she may achieve her goal in a different degree (persuade with a different strength). The second problem is how can we formally model uncertainty? Although a great deal of interest has focused recently on logics of beliefs, there has been relatively little work on providing formalisms for representing the gradation of beliefs. Despite engaging various mathematical or logical methods, the task is still viewed as very difficult and unsatisfactorily accomplished. In the paper, we analyze three widespread techniques concentrating particularly on a question: which one of them is capable to capture shades of uncertainty in the most appropriate way to describe a persuasion process.

The rest of the paper is organized as follows. In Section 2, we introduce the **model of gradation** of beliefs specified for a persuasion theory. In Section 3, we analyze three possible methods of **formalizing** this model, namely: multi-valued, probabilistic and graded modalities.

2 The model of beliefs gradation

We start with investigating the issue of modeling the gradation of beliefs. First we demonstrate the general background of how we understand the role of degrees of uncertainty in convincing.

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[3] See the webpage http://perseus.ovh.org/ for more details.

Then we formulate requirements which should be satisfied by the model of uncertainty such that it adequately represents the phenomena characteristic for a persuasion process.

2.1 Degrees of beliefs in a persuasion process

In real-life practice we deal not only with the type of convincing which changes people's opinion totally from "yes" to "no" (or the opposite way), but also with the kind of persuasion that aims at **influencing the degree** of beliefs. Indeed, throughout the process of persuasion the belief-attitudes are not only black or white ("It is *for sure* true", "It is *for sure* false"), but they represent various shades of uncertainty such as "I am *almost sure* that this is true", "The thesis is *rather* true", "*Maybe* you are right", "It *seems* to be false", etc.

What phenomena of convincing can we capture when we refer to the gradation of beliefs? First of all, it plays an important role for describing **persuasiveness** of particular persuaders or arguments. That is, various agents or means of convincing may bring about different *effects* with respect to a given audience and have different *power* in succeeding (the audience may become convinced in a various degree). Consider as an example the following statements: "I chose the lesser of two evils when voting for the party X", "The truth is that I didn't vote for X - I rather voted against the party Y". We may understand that the speaker was not absolutely certain whether X's electoral program was good, but she voted for X anyway. This means that the party did not persuade the person in a high degree, however it managed to achieve the higher grade than the other party did (X was more persuasive than the rival). Associating the value of 0 with an attitude of "absolutely against" and 1 with "absolutely for", we could interpret above statements as follows: the speaker is convinced to X in the degree lesser than 0.5, but still higher than the degree assigned to Y's opinions. In fact, in practice the rivals may be so weak that the grade assigned to party's reputation may be pretty low, but you vote for it anyway.

Moreover, with the help of graded beliefs the effect of **arguments' accumulation** may be studied. Under this notion we mean that adding the arguments may consecutively *strengthen* the power of persuader's success. When we were able to utter only non-graded beliefs, then we could capture just the change of beliefs of the black or white character, i.e. from "convinced" to "unconvinced" or from "unconvinced" to "convinced". In such a model we would not be able to express the change of the degree of beliefs resulting from adding more arguments. In real-life practice the persuader sometimes continues with convincing even though he succeeded just to achieve the stronger success. For example, the party X may decide to keep persuading (i.e. proceed with its election campaign) not only for it must defend itself against Y's attacks, but for new arguments could bring about the stronger success for X.

2.2 Requirements for the model of graded beliefs

Now, we want to specify what properties our model of graded beliefs should satisfy in order to apply it in a persuasion theory. The postulated requirements are as follows:

R1 *Gradation expressed explicitly in a language.*

The aim of the persuasion is to influence the other person to adopt a particular belief. However, as we mentioned above, adopting a thesis does not necessarily mean to accept it without a doubt. Sometimes it is enough just to change the degree of beliefs. Surprisingly, not always to a higher one. It could be a case in which the persuader wants to discourage someone from something. So, we assume that persuasion is **successful** when a persuaded person believes the thesis with a degree which satisfies a person who proposed it. Therefore, in order to specify a property which expresses a success we need to be able to say that a degree of someone's beliefs is changed, but first and foremost we have to be able to identify what such a degree is. For this reason, we want to describe grades of beliefs directly in a language of a logic. That is, the degree should be expressed explicitly in the syntax of a belief formula unlike a logical value of this formula which is determined by its semantics. This is why we choose only the numerical methods of modeling uncertainty instead of the symbolic approaches like non-monotonic logics (see e.g. [2, 10] for comparison of these two techniques). For example, a formula which expresses that an agent i believes a thesis α with a degree d_1 could be of the form $B_i^{d_1}\alpha$. Then, if after persuasion process this grade changes to a new one, say d_2, we will write $B_i^{d_2}\alpha$. Observe that in this place we disregard how these degrees are computed (we will go back to this topic).

R2 *Nesting belief operators of different degrees.*
Consider a situation in which Peter wants to convince Susan to spend their holidays at the seaside. He tries various tactics and gives different arguments, e.g. "a view of the sea is beautiful", "we can sunbath and swim", "I know a nice and cheap place where we can stay", etc. The question is when will he achieve the success? The immediate answer is: when Susan adopts his idea. But how can he learn about it? Notice that Peter will think he is successful only if Susan tells him or shows that she accepts his proposal. More precisely, the persuasion will end with **subjective success** when Peter will believe that Susan believes they should go to the seaside. As we described before, it is sufficient that Peter will believe with a degree, say 1, that Susan believes with a degree, say $\frac{3}{4}$, that spending holidays at the seaside is a good idea. In another situation Peter can intend to convince Susan that Cyprus is not a good place for vacation. In this case he will be contented if he discourages Susan from going to this island, i.e., formally speaking, if Peter will believe with a degree, say $\frac{3}{4}$, that Susan believes with a degree, say $\frac{1}{4}$, that Cyprus is a nice place for holidays. Therefore, in order to describe this kind of success in persuasion we need to nest belief modalities with different grades. This is the next requirement which the sought-after logic should fulfill. Indeed, it is not enough to affirm that the persuaded person is convinced of something. The other important issue is whether the persuader is sure (or rather sure) of his success.

R3 *Expressing the ratio of a number of states verifying thesis to a number of states refuting it.*
One of the most important questions becomes: how should we define the beliefs gradation in the semantics? Let us start with **non-graded approach** of representing agents' cognitive attitudes which is commonly employed in artificial intelligence and computer science. In this approach, formulas expressing beliefs of agents are interpreted in the Kripke-style semantics. Given a set of *propositional variables* V_0 and a set of *agents* $Agt = \{1, 2, \ldots, n\}$ (in fact, it is a set of names of agents marked by natural numbers), the semantical model is defined as a tuple $\mathcal{M} = (S, RB_1, \ldots, RB_n, v)$ where:

- S is a set of *states* (or *possible worlds*),
- $RB_i \subseteq S \times S$ is an *accessibility relation* for each agent $i \in Agt$,
- $v : S \to \{\mathbf{1}, \mathbf{0}\}^{V_0}$ is a function which assigns to every state a *valuation* of propositional variables.

In a standard doxastic logic it is possible to express three types of belief attitudes: $B_i\alpha$ - which intuitively means that an agent i is absolutely sure that α, $M_i\alpha$ - an agent i allows α to be true, and $N_i\alpha$ - an agent i is neutral with respect to a logical value of α. Formally:

$$\mathcal{M}, s \models B_i\alpha \text{ iff for every } s' \in S \text{ if } (s, s') \in RB_i \text{ then } \mathcal{M}, s' \models \alpha.^{4}$$

The other modalities are derived in the following way: $M_i\alpha \leftrightarrow \neg B_i\neg\alpha$ and $N_i\alpha \leftrightarrow \neg B_i\alpha \wedge \neg B_i\neg\alpha$.

We can capture the idea nicely considering some example. The states can represent, roughly speaking, different possible "versions" or "images" of the reality. Obviously, only one of such versions may be actual, but an agent is not certain which one. Say that in Fig. 1a-1d the state s represents the reality. However, an agent i is unsure of how it actually looks like. Therefore she imagines the reality in different ways. In Fig. 1a, i considers three options as the so-called *epistemic alternatives*: s_1, s_2 and s_3. The accessibility relation RB_i represents what options are allowed by i as these alternatives. In 1a for example, the relation links the real state s with those which according to the agent are the conceivable shapes of the reality (i.e. with s_1, s_2, s_3).

Now, assume that a persuader wants to convince the agent i to a thesis α. According to non-graded semantics an individual believes only this what is true in *every* state that she considers possible (every state accessible through transitions in her relation). Thus, in Fig. 1a at the state s the agent i disbelieves the thesis ($s \models B_i\neg\alpha$) while in Fig. 1d at state s she believes it ($s \models B_i\alpha$). Observe that despite some differences between 1b and 1c, in both of those situations i is

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[4] The full language is the propositional calculus extended with belief modalities. The semantics of propositions and boolean connectives are defined in the standard manner.

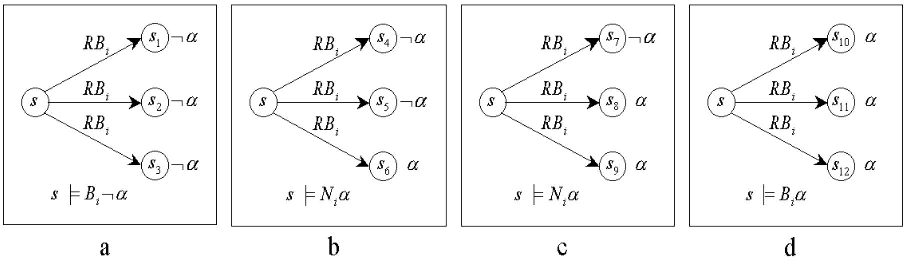

Fig. 1. The differences in an agent i's cognitive attitudes towards a thesis α.

neutral with respect to the thesis (she believes neither α nor $\neg\alpha$). As a result, those cases are undistinguishable in non-graded approach.

Once we want to consider the persuasion's influence on agent's beliefs, we may express the following types of changes here: * from the *negative* opinion about a thesis (like in Fig. 1a) to the *neutral* one (1b or 1c), * from the *negative* (1a) to the *positive* opinion (1d), * from the *neutral* (1b or 1c) to the *positive* belief (1d) (or the opposite directions). This seems to be not enough. The mentioned cases 1b and 1c are the demonstrative examples of such an insufficiency since they show that the persuasion **influences the grades** of beliefs to some extent - in 1c i accepts the thesis in a higher degree than in 1b. The change is even more essential than that. Notice that in 1c she stakes more on the thesis than on the antithesis (while in 1b - inversely) for a greater number of i's epistemic alternatives verifies α. In some specific circumstances, it could mean the success for the persuader. However, this cannot be expressed in non-graded approach. Thus, from a viewpoint of capturing differences in the persuasiveness we would like to have more expressible framework at our disposal. It should extend non-graded approach in the direction which allows us to describe e.g. how strong an agent believes a thesis or whether she is more certain it is true than false. Simply stated, we expect that in our model of graded beliefs we could somehow compare the number of i accessible states verifying thesis to the overall number of states considered by agent i. It will enable to differentiate the degree of belief in 1c (i.e. the ratio of $\frac{2}{3}$) from the degree in 1b ($\frac{1}{3}$).

R4 *Distinguishing the likelihood of epistemic alternatives (weights).*
In real-life practice, the epistemic alternatives are not always treated as **equally likely**. Imagine two players Kasia and Magda, each drawing one card from a stack of three cards {Ace, King, the Ten}. Assume that Kasia holds King and she tries to guess what a card Magda has. She imagines the reality in two versions - s_1 and s_2. According to s_1 Magda holds the Ten and according to s_2 she holds Ace. In standard doxastic semantics, we say that both of these states are accessible by means of Kasia's doxastic relation (see Fig. 2a). Observe that Kasia considers both of those versions as having the same chances. The question becomes: how are we going to model the case when Kasia evaluates s_2 as more likely version of the reality since it seems to her that Magda is lucky at cards? The solution would be to assign weights to particular transitions in an accessible relation. In this manner, we could relate greater certainty with the state s_2, say 0.8, than with s_1, say 0.2 (see 2b).

In the semantics of graded beliefs, the weights should be included if we wanted persuasion to change someone's belief in the likelihood of particular visions of the reality (i.e. to **change the weights**). Say that initial moment of convincing is like in Fig. 2b and Magda intends to influence Kasia to increase her belief in the state s_1, i.e. to achieve the case in which Kasia's beliefs are such as depicted in Fig. 2c. In this way, she aims to convince Kasia to believe that she does not have a winning card. Observe that in "non-weights" approach the relation may also be viewed as labeled, but only by the values 0 and 1 (where 0 means "inaccessible via the relation" and 1 "accessible via the relation"). In this sense, marking the transitions with graded values is a natural extension for "non-weights" framework.

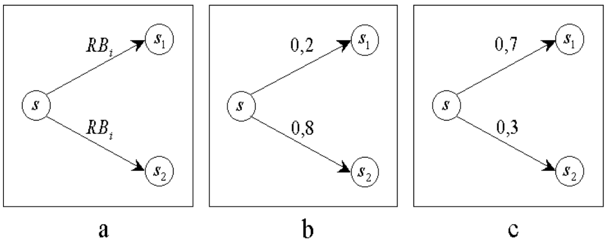

Fig. 2. The differences in the likelihood of an agent i's visions of the reality.

R5 *Sound and complete axiomatization.*
The last requirement concerns metamathematical properties of logical systems. First of all, a formalism which we can employ to **reason about a persuasion** should have precise syntax and well defined semantics compatible with intuition that we have about beliefs of people and software agents. Furthermore, since we want to use such a logic for specification and verification of properties of distributed systems like multi-agent systems it must have sound and complete axiomatization. This means that it should offer a set of axioms and inference rules such that every provable formula in this deductive system is valid with respect to a given class of structures and every formula that is valid with respect to this class of structures is provable.

Considering a system with n agents, the axiom system for the **non-graded logic** of beliefs consists of the following axioms (for $i \in Agt$):

A0 All propositional tautologies
K $(B_i\alpha \wedge B_i(\alpha \rightarrow \beta)) \rightarrow B_i\beta$ (distribution of beliefs)
D $B_i\alpha \rightarrow \neg B_i\neg\alpha$ (consistency of beliefs)
4 $B_i\alpha \rightarrow B_iB_i\alpha$ (positive introspection)
5 $\neg B_i\alpha \rightarrow B_i\neg B_i\alpha$ (negative introspection)

and derivation rules: $\frac{\alpha,\quad \alpha\rightarrow\beta}{\beta}$ (Modus Ponens), $\frac{\alpha}{B_i\alpha}$ (Necessitation).

Let us give some intuitions. The axiom K says that beliefs of an agent i are closed under classical logical consequence, i.e., if the agent believes α and believes that α implies β then she also believes β. So, she knows all logical consequences of her beliefs. Next, the axiom D postulates that beliefs of an agent cannot be contradictory with each other. Thus, if the agent believes a thesis then she disbelieves the antithesis. The axiom 4 states that an agent believes that she believes something. This means that the agent is aware of her beliefs. The axiom 5 expresses that an agent believes that she does not believe something, i.e., she is conscious of her ignorance.

The above system, called weak-S$5_{(n)}$ is commonly employed to capture beliefs of agents in distributed and intelligence systems. If we add to this system the axiom T (for $i \in Agt$):

T $K_i\alpha \rightarrow \alpha$ (infallibility of knowledge)

then we obtain a system commonly used for modeling knowledge, called S$5_{(n)}$. The axiom T affirms that known facts are true. In other words, the knowledge of agents cannot conflict with reality.

It is proved (see e.g. [5]) that the system weak-S$5_{(n)}$ is a complete axiomatization of the class of all Kripke models with n agents with accessibility relations that are serial, transitive, and euclidean, while the system S5_n refers to the class of Kripke models with the equivalence relations.

3 Possibilities of formalization

Now, we analyze the possibilities of formalizing the model of the gradation of beliefs characterized in the previous section. We discuss multi-valued logic of B. Konikowska and W. Penczek [8], probabilistic modal system of R. Fagin, J.Y. Halpern and N. Meggido [4, 3] and finally - graded modalities' formalism interpreted epistemically by W. van der Hoek and J.J. Meyer [9, 11–13].

Moreover, we evaluate them from the point of view of formulated requirements focusing thereby on the appropriateness of the description of a persuasion process.

3.1 Multi-valued representation

Multi-valued logic seems to be a particularly appropriate framework for modeling uncertainty of opinions. Although there are many systems which explore more than two logical values, there is relatively little research on connections between multi-valued formalisms and reasoning about degrees of beliefs. An interesting approach to representing knowledge is multi-valued μK-calculus (mv μK) presented by Beata Konikowska and Wojciech Penczek in [8]. It is a very expressive logic which allows specifying knowledge and time in multi-agent systems. The main aim of [8] is to show a model checking technique which can be used for verifying properties involving multivalued μK-calculus or its subsets. However, we focus only on the part concerning modeling knowledge.

For expressing knowledge properties of multi-agent systems in mv μK the *epistemic modality* K_i is explored. We read it as follows: K_i - "agent i knows". This formula is interpreted in a multi-valued Kripke structure. Given a set of agents Agt and a set of propositional variables V_0, by a model we understand a tuple $\mathcal{M} = (S, \mathcal{R}, \sim_1, \ldots, \sim_n, v, \mathcal{L})$ where

- S is a finite state of global *states* of the system,
- $\mathcal{R} : S \times S \to L \times L$ is the multi-valued *transition relation* on S,
- $\sim_i \subseteq S \times S$ $(i \in Agt)$ is an *epistemic accessibility relation* for each agent $i \in Agt$ defined by $s \sim_i s'$ iff $l_i(s') = l_i(s)$, where $l_i : S \to Loc_i$ extracts the local state of agent i from a global state s; observe that $\sim_i$ is an equivalence relation,
- $v : S \to L^{V_0}$ is a *valuation function* for propositional variables,
- $\mathcal{L} = (L, \leq, -)$ is a De Morgan algebra.[5]

We quote the semantics only for the epistemic modality:

$$[K_i\alpha]^{\mathcal{M}}(s) = \bigcap_{\{s' \in S | s \sim_i s'\}} [\alpha]^{\mathcal{M}}(s').$$

Now, we give an example to show more intuitions concerning the operator K. Consider the agent 1 working for the bank, whose task is to decide if a customer applying for a loan, call her agent 2, is to be granted it or not. For this purpose, the agent 1 checks the reliability of the customer in different databases which contain information about bad debtors. Let p be a proposition expressing that "The agent 2 is a good debtor". Furthermore, assume that at a state s_1 the agent 1 considers as its epistemic alternative four states s_1, s_2, s_3, s_4 (i.e. $s_1 \sim_1 s_j$ for $j = 1, 2, 3, 4$) with the following valuation for p: $v(s_1, p) = \frac{3}{4}$, $v(s_2, p) = 1$, $v(s_3, p) = \frac{3}{4}$, $v(s_4, p) = \frac{1}{4}$. The value 1 means that according to the databases the agent 2 is a good debtor for sure, $\frac{3}{4}$ - she is rather a good debtor, $\frac{1}{4}$ - she is rather a bad debtor. According to the semantics presented above, the value of the formula $K_1 p$ at the state s_1 is $\bigcap_{s \in \{s_1, s_2, s_3, s_4\}} [p]^{\mathcal{M}}(s) = \frac{3}{4} \cap 1 \cap \frac{3}{4} \cap \frac{1}{4} = \frac{1}{4}$. As a result, we say that the agent 1 knows that the agent 2 is rather a bad debtor.

Observe that within this approach the **requirement R1** postulated in section 2.2 is not fulfilled since the gradation of epistemic formulas is hidden in their semantics and not indicated directly in their syntax. It is a serious limitation since we can assign a logical value to e.g. a formula $K_i\alpha$, but we cannot say with what a degree an agent i knows α. Consequently, nesting of epistemic formulas with different grades is not possible what fails to satisfy **R2**. Again, we can evaluate a formula e.g. $K_i K_j \alpha$, but there is no way to express that an agent i knows with a degree d_1 that an agent j knows with a degree d_2 that α holds.

What is even worse, the logical values cannot be understood as *degrees* of knowledge what dissatisfies **R3**. Let us give some more explanations. In multi-valued logics there is a function which assigns one of the many values to each propositional variable. In this way it is possible to express that some thesis is true, rather true, rather false, false etc. and thereby gradate its truthfulness. For example, we can say "today is warm", "today is rather warm", "today is rather

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[5] We don't discuss the model in detail since it isn't essential for our study. For more explanations see [8].

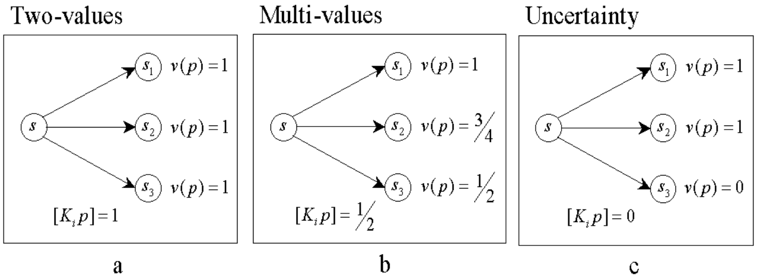

Fig. 3. The logical values of formulas expressing an agent i's knowledge.

not warm" etc. This function can be extended to give the gradation of all formulas, in particular, the epistemic ones. For example, we can determine the logical value of a formula K_ip (say it is $\frac{1}{2}$). So, we can express that "the logical value of a statement "an agent i knows p" is $\frac{1}{2}$". Notice that the value $\frac{1}{2}$ encodes the information about what is a degree of the thesis p from the point of view of the agent i and not the information of a degree of uncertainty of this agent about the thesis. The value $\frac{1}{2}$ of a formula K_ip means merely that in all epistemic alternatives of the agent i the logical value of p is *not less* than $\frac{1}{2}$ (see Fig. 3b). In fact, it is the natural extension of two-valued approach where the value 1 of a formula K_ip means that in all worlds that the agent i considers possible the value of p is *not less* than 1, i.e. p is true in all i's accessible worlds (see Fig. 3a). What is more, notice that assuming that the agent is *uncertain* and considers two states in which p is true and one state in which p is false, in both of two valued and multi valued approach the logical value of the formula K_ip is 0 (see Fig. 3c). Consequently, in this logics it is not possible to determine the ratio of the number of i's epistemic alternatives in which p holds to all i's epistemic alternatives, what makes the requirement R3 not fulfilled.

Return to the example of granting the loan. We can express that the agent 1 knows that the statement "the agent 2 is a good debtor" has the value $\frac{1}{4}$. In other words, the agent 1 is *absolutely sure* that the agent 2 is *rather* not a good debtor. Clearly, it is not the same so to say that the agent 1 is *rather sure* that the agent 2 is a good debtor. For this reason, in multi valued approaches we can gradate various theses but not knowledge of agents. Similarly, consider a situation in which Ann and Susan want to go to a nice restaurant and Ann says "I am rather sure that there is a wedding party in this restaurant tonight and we cannot eat a dinner there". Such a statement is not expressible in the multi valued logic. We could only say "I am (absolutely) sure that there is rather a wedding party in this restaurant". There is no need to explain that it is senseless if we refer with "rather" to the reality and not to beliefs, i.e., when we want to talk about whether the wedding party actually takes place in the restaurant. Indeed, there is a big difference between grades of truthfulness of assertions expressible in multi valued frameworks and grades of knowledge about which we would like to reason. As a result, multi valued approach cannot be adapted for reasoning about shades of agents' uncertainty, and thereby changes of degrees of beliefs during a persuasion process.

Finally, notice that in Multivalued Logic of Knowledge and Time there is no way to show that some of agents' epistemic visions of the current global state in which they are, can occur with different likelihood. So, the requirement **R4** is missed. Furthermore, in [8] an axiomatization of presented system is not given or discussed. Thus, the requirement **R5** is also not satisfied.

3.2 Probabilistic modal logic

Probability also seems to be a very intuitive and natural tool to describe gradation of beliefs. The probabilistic modal logic presented here is called AX_{MEAS} and was introduced in 1990s by Ronald Fagin, Joseph Y. Halpern and Nimrod Meggido in [4] and further extended by the first two authors in [3]. It combines the probability theory with Kripke-style semantics by placing the probability measure on the set of *possible worlds*. Let us start with a chosen elements of its syntax. A basic **probability formula** is an expression: $w_i(\alpha) \geq b$ where b is a rational number and $i \in Agt$ is an agent. The formula expresses i's graded belief in α and intuitively means "according to i, formula α holds with probability at least b". Moreover, some useful abbreviations are introduced, e.g. $w_i(\alpha) = b$ for $(w_i(\alpha) \geq b) \wedge (w_i(\alpha) \leq b)$. Assume that a degree of i's belief about Tweety's ability to fly is greater than 0.6. It may be expressed in the language of AX_{MEAS} in the following way: $w_i(p) > 0.6$ where w_i stays for "probability according to the agent i" and p for "Tweety flies".[6]

The formulas are interpreted in the **probability structure** $\mathcal{M} = (S, P, v)$ where S is a set of states (or possible worlds), v is a function which assigns to every state a valuation of propositional variables $v : S \rightarrow \{\mathbf{1}, \mathbf{0}\}^{V_0}$ and P is a *probability assignment* which assigns to each agent $i \in Agt$ and state $s \in S$ a *probability space* $P_{i,s} = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$ where

- $S_{i,s} \subseteq S$ (called *sample space*),
- $\chi_{i,s}$ is a σ-algebra of subsets of $S_{i,s}$ (called *measurable sets*), i.e., a set of subsets of $S_{i,s}$ containing the empty set and closed under complementation and countable union,
- $\mu_{i,s}$ is a **probability function** on the measurable sets $\mu_{i,s} : \chi_{i,s} \longrightarrow [0, 1]$, i.e. $\mu_{i,s}$ is a mapping from $\chi_{i,s}$ to the real interval $[0, 1]$.

Intuitively, the probability space $P_{i,s}$ shows agent i's probabilities on events, given that the state is s. Assuming that $\mathcal{M}, s \models \alpha$ is inductively defined, we may associate with each formula α the set of the states in which the formula is true and which belongs to $S_{i,s}$, i.e. $S_{i,s}(\alpha) = \{s' \in S_{i,s} : \mathcal{M}, s' \models \alpha\}$. Given a probability structure $\mathcal{M}$ and a state s, we define the **semantics of the probability formulas** of AX_{MEAS} as follows:

$$\mathcal{M}, s \models w_i(\alpha) \geq b \text{ iff } \mu_{i,s}(S_{i,s}(\alpha)) \geq b.$$

This needs some explanation. Recall that according to the Kripke-style semantics the states represent different possible "visions" of reality. In non-probabilistic approach, the individual believes that Tweety flies only when it is true in *every* state she considers as her epistemic alternatives. The idea for adding the probability is that a formula can be true only in *some* of such states and the (graded) belief is generated anyway. Thus, "Tweety flies" would hold in some states possible from the i's viewpoint, and not in others. Given a model $\mathcal{M}$ and a state s, an agent i believes that Tweety flies with a degree greater than 0.6 when according to i's probability assignment at s, the set of worlds where "Tweety flies" holds (i.e. the set $S_{i,s}(p) = \{s' \in S_{i,s} : \mathcal{M}, s' \models p\}$) has the probability measure greater than 0.6. Formally, we shall write: $\mathcal{M}, s \models w_i(p) > 0.6$ iff $\mu_{i,s}(S_{i,s}(p)) > 0.6$. Say that the probability structure is a tuple $\mathcal{M} = (S, P, v)$ where $S = \{s, s_1, \ldots, s_{10}\}$ and s represents the reality. Let us associate with an agent i and a state s (by means of an assignment P) the sample space consisting only of s and s_1 with s and s_1 both being measurable and having measure 0.3 and 0.7, respectively (see Fig. 4). That is, the probability space is $P_{i,s} = (S_{i,s}, \chi_{i,s}, \mu_{i,s})$ such that $S_{i,s} = \{s, s_1\}$, $\chi_{i,s} = \{\emptyset, S_{i,s}, \{s\}, \{s_1\}\}$ and $\mu_{i,s} = \{\langle \emptyset, 0\rangle, \langle S_{i,s}, 1\rangle, \langle \{s\}, 0.3\rangle, \langle \{s_1\}, 0.7\rangle\}$. Further, assume that the sentence "Tweety flies" is false in s and true in s_1, i.e. $v(s)(p) = \mathbf{0}$ and $v(s_1)(p) = \mathbf{1}$. Now, is it a case that i's degree of belief about Tweety's ability to fly is greater than 0.6 in a state s? We have $S_{i,s}(p) = \{s' \in S_{i,s} : \mathcal{M}, s' \models p\} = \{s_1\}$. The measure of this set is greater than 0.6 since it equals to 0.7. Thus, we have $\mu_{i,s}(S_{i,s}(p)) > 0.6$, so $\mathcal{M}, s \models w_i(p) > 0.6$.

They assume the following **axioms for reasoning about probabilities**:

W1 $w_i(\alpha) \geq 0$ (nonnegativity)
W2 $w_i(\mathit{true}) = 1$ (the probability of the event *true* is 1)
W3 $w_i(\alpha \wedge \beta) + w_i(\alpha \wedge \neg\beta) = w_i(\alpha)$ (additivity)
W4 $w_i(\alpha) = w_i(\beta)$ if $\alpha \leftrightarrow \beta$ is a propositional tautology (distributivity)

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[6] Sometimes, they do not make a straightforward connection between probability and degrees of beliefs. However, some of the articles (e.g. [7]) show that they identify these two notions.

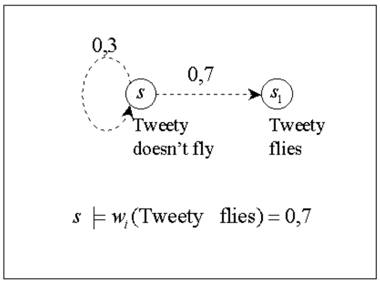

Fig. 4. An agent i's degree of belief about Tweety's ability to fly.

The axiomatization proposed for this logic is proved to be sound and complete (see [4] or [3] for proofs of soundness and completeness as well as a full list of axioms for the logic). The axioms for reasoning about probabilities are simply a translation of the standard axioms for probability in finite domains to the language of AX_{MEAS}. The axiom W1 assumes that a degree of i's belief is at least 0 and W2 - that a degree of i's belief about the event *true* is 1. W2 and W3 correspond precisely to the axioms that characterize probability measures in finite spaces. The axiom W3 says the probability measure is finitely additive (but not countably additive). The axiom W4 expresses that for an equivalence which is a propositional tautology, i believes its arguments with the same degree.

Needless to explain that this logic fulfils the **requirements R1-R2** formulated in section 2.2. Moreover, observe that an *accessibility relation* assumed in the standard doxastic semantics is here replaced with a *probability measure*. As a result, this logic is capable of modeling the situations in which an agent considers some worlds to be more likely than others. This means that the presented technique satisfies the condition **R4**. Recall the example of Kasia and Magda playing cards. Kasia holds King and guesses what card Magda has - the Ten (the state s_1) or Ace (s_2). By means of probability measure we label the sets of states with different values. In this way, we may assign greater certainty with the state s_2 (we assumed the value 0.8) than with s_1 (0.2) since Kasia thinks that Magda is lucky at cards. Thus, in the state s_1 we shall have $\mu_{Kasia,s_1}\{s_2\} > \mu_{Kasia,s_1}\{s_1\}$. Observe that although the requirement **R3** is not satisfied here, the measures offer the similar effect of expressing the gradation of beliefs. Furthermore, note that within the framework the sum of these weights should equal exactly 1. As a result, two different sets of worlds cannot have measures which are relatively high (e.g. 0.6) at the same time.

This approach has one serious limitation related to the requirement **R5**. The axiomatization of AX_{MEAS} *differs substantially* from the ones commonly used to give the formal description of knowledge and beliefs of agents in distributed and intelligent systems. Clearly, the inconvenience of the unconventional notation is not the biggest problem here. The more important is that the attributes of beliefs assumed by this logic are not so well studied as it is done for standard epistemic or doxastic systems. For example, it is not obvious if the positive or negative introspection is postulated in AX_{MEAS} (as we noted in Section 2.2, these properties are assumed in the standard approach). Some more careful research is required to be conducted here.

We want to give one more **comment** here. In [3], Fagin and Halpern introduce some essential extension with respect to the prior versions of this logic (e.g. in [4]). Without these modifications, the probabilistic modal approach would be completely useless for a persuasion theory (although they did not mean such an application). That is, a probability function is here related to an agent and a state. Associating different measures with each *agent* allows individuals to vary in opinions what is absolutely necessary if we want to express a conflict between them. As long as everyone share the same beliefs, there is no need to persuade others. In turn, relating probability to a *state* allows an individual to have different opinions depending on circumstances. It is important since the result of my persuasion depends on the initial state that I choose for giving arguments. I may succeed if I sense the good moment or fail if I start when my audience has the strong negative attitude resistant to anything I say. In the authors' prior papers, the probability structure is defined as the tuple $\mathcal{M} = (S, \chi, \mu, v)$ where S is a set of states, χ is a σ-algebra of subsets of S, μ is a probability function $\mu : \chi \longrightarrow [0, 1]$, and v is a valuation function. Say that the set $\{s_1\} \subseteq S$ has a measure 0.6. Clearly, in such a case the measure is the same for each agent and each state. That is, since there is only one probability function all individuals must share beliefs of the same degrees (in particular, since $\mu\{s_1\} = 0.6$ for each $i \in Agt$) no matter at what state ($\mu\{s_1\} = 0.6$ for each $s \in S$).

3.3 Graded modalities' formalism

The last framework discussed here is called Epistemic Logic of Graded Modalities (Gr(S5)) and is particulary interesting since it is elaborated in modal epistemic logic itself [11–13, 9]. In the formalism it is expressible that an agent accepts a thesis α although she is conscious of some exceptions to α. Thus the logic with graded modalities allows to deal with types of knowledge that are less absolute than in traditional approach. Recall that in non-graded framework an agent i knows α if α holds in *all* states she considers as possible. However, in some situations it might be desirable to be able to express that the agent has more confidence in α than in $\neg\alpha$. The logic with graded modalities provides a solution to this problem by adding **quantitative modalities** M^d and K^d ($d \in \mathbb{N}$) enabling to describe the agent's point of view in a more precise manner. Intuitively, we understand the formula $M_i^d\alpha$ as follows: "agent i accepts α iff there are *more* than d accessible states verifying α". In the same spirit, dual formula $K_i^d\alpha$ is true iff at *most d* accessible states refute α. This logic can be employed in multi-agent systems where there are different sources to judge the same proposition and agents have to make a decision on their basis. It may happen that results obtained in some sources are false because of faulty sensors or bad calculations. In such cases it is understandable that some data could be refused.

The formulas are interpreted in **Kripke structure** $\mathcal{M} = (S, RK_1, \ldots, RK_n, v)$, where S is a set of worlds, RK_i ($i \in Agt$) a binary relation on S, $v : S \rightarrow \{\mathbf{1}, \mathbf{0}\}^{V_0}$ a truth assignment (recall that V_0 is a set of propositions). It is assumed that RK_i is an equivalence relation. For a Kripke structure $\mathcal{M}$ the truth of a formula $M_i^d\alpha$ at $s \in S$ is defined as follows:

$$\mathcal{M}, s \models M_i^d\alpha \text{ iff } |\{s' \in S | (s, s') \in RK_i \text{ and } \mathcal{M}, s' \models \alpha\}| > d \ (d \in \mathbb{N}).$$

Here $|X|$ stands for the cardinality of the set $X \subseteq S$. A formula $K_i^d\alpha$ is an abbreviation for $\neg M_i^d\neg\alpha$. We use also $M!_i^d\alpha$ where $M!_i^0\alpha \Leftrightarrow K_i^0\neg\alpha$, $M!_i^d\alpha \Leftrightarrow M_i^{d-1}\alpha \wedge \neg M_i^d\alpha$, if $d > 0$. From the definition above, it is clear that $M!_i^d$ means "*exactly d*".

The system GR(S5) has two inference rules Modus Ponens and Necessitation: $\frac{\alpha}{K_i^0\alpha}$ and the following schemes of **axioms** (for each $d, d' \in \mathbb{N}$ and $i \in Agt$):

A0 all propositional tautologies
A1 $K_i^0(\alpha \rightarrow \beta) \rightarrow (K_i^d\alpha \rightarrow K_i^d\beta)$
A2 $K_i^d\alpha \rightarrow K_i^{d+1}\alpha$
A3 $K_i^0\neg(\alpha \wedge \beta) \rightarrow ((M!_i^d\alpha \wedge M!_i^{d'}\beta) \rightarrow M!_i^{d+d'}(\alpha \vee \beta))$
A4 $\neg K_i^d\alpha \rightarrow K_i^0\neg K_i^d\alpha$
A5 $K_i^0\alpha \rightarrow \alpha$

This system is the graded modal analogue of the modal system S5. The axiom A1 is a kind of generalized K-axiom, A2 and A3 describe ways to decrease and increase grades in the possibility operators. A4 is an equivalence of the negative introspection axiom 5. Finally, A5 expresses that known facts with no exceptions are true what corresponds to T-axiom.

The first advantage of this formalism is that degrees of knowledge are employed *directly in the language* what fulfils the **requirement R1**. Remind that the formula $K_i^d\alpha$ means that among all epistemic alternatives which an agent i considers possible there are d states in which α is not satisfied. Note that the higher the degree d in K operator is, the less certain the knowledge is. Such kind of uncertainty is used in systems working on computers which have multiple processors and where knowledge that is not absolutely true in all input sensors is used to make decisions. Moreover, specification of properties which require *nesting modalities* with different grades is possible what satisfies **R2**. So, we can formally express that an agent 1 considers at most 10 states which refute that an agent 2 leans towards a thesis (i.e. considers at most 2 states which refute a thesis α): $K_1^{10}K_2^2\alpha$.

Once we consider the requirement **R3**, it may seem at first that this approach can be reduced to the probabilistic framework by determining the ratio of all epistemic alternatives where a thesis holds to all epistemic alternatives of a given agent. However, it is not. For example, let us assume that an agent 1 is aware of exactly 10 possible states in which a thesis α holds, i.e. $M!_1^{10}\alpha$. Notice that in such a case it is still not clear whether the agent believes stronger in α than in $\neg\alpha$ since we do not know how many states he considers as his epistemic alternatives. If there are 10 i-accessible states then the agent should accept the thesis while if there are 100 states she should refuse it. However, it is reflected neither in the formula $M!_1^{10}\alpha$ nor in the formula $M_1^{10}\alpha$. To complete the information we propose to consider a new formula $M!_1^{10,100}$ which is an equivalent to the conjunction $M!_1^{10}\alpha \wedge M!_1^{100}true$. It says in how many i's epistemic alternatives the thesis is true and how many states the agent considers as possible.

Actually, this does not improve much the situation because it is not possible to compare directly in the language the numbers of states in which the thesis holds with the number of all accessible states. What we mean here is that although $\frac{10}{100} = \frac{1}{10}$ the formulas $M!_1^{10,100}\alpha$ and $M!_1^{1,10}\alpha$ are not equivalent. The first one says that there are 100 states and 10 of them satisfy α while the second formula expresses that there are 10 states and only one satisfies α. The difference is obvious. Consequently in this approach there is no way to compare *ratios* of the number of states which are considered by an agent and verify a thesis to the number of all states considered by the agent. Preferred arithmetical calculations could be done only in a metalanguage. The other inconvenience is that using formulas of the language of GR(S5) it is very difficult to specify some properties of multi-agent systems. Assume that we would like to check whether an agent 1 is able to convince an agent 3 that an agent 2 is a good debtor with a degree of satisfaction equal to $\frac{3}{4}$. The question is what a formula should be true after the argumentation process? Notice that assuming that p means "agent 2 is a good debtor", the success is expressed by formulas $M!_3^{3,4}p$, $M!_3^{6,8}p$, $M!_3^{9,12}p$ etc. So, which of them should we choose? Or maybe verification of all of them can give the answer? Furthermore, observe that a formula $M!_3^{4,4}p$ does not express the success in spite of the fact that an agent 3 is absolutely sure that p holds. What is worse, if it is a case that an agent has only 5 epistemic alternatives the success can not be achieved and expressed at all. Therefore, the requirement R3 is not fully satisfied.

To evaluate a formula $K_i^d\alpha$ we consider the epistemic relation which shows in how many accessible states a thesis α holds. Clearly, it is assumed all those states are *equally likely*. However, there are situations in which we would like to differ their chances like we did in Section 2.2 when Kasia and Magda played cards. Formally it could be reflected in values assigned to each couple $(s, s') \in R_i$ $(i \in Agt)$. Unfortunately, in Gr(S5) it is not taken into consideration. Thereby, like in multi-valued approach, the requirement **R4** is not fulfilled.

Despite of some weaknesses, this approach has also a strong advantage. As we noted above the logic GR(S5) is an equivalent of modal S5 system. Thus its axioms are much more *intuitive* than those of probabilistic modal logic described in Section 2.2. This system is also complete what is proved in [6]. Thereby, the requirement **R5** of our model for persuasion is satisfied. Further, due to its similarity to the standard approach it is relatively easy to transform the epistemic logic with graded modalities into a *doxastic* one (what is particularly important since persuasion refers to beliefs and not knowledge). It is sufficient to loosen the assumption that the accessibility relation is an equivalence relation and to require that it is serial, transitive, and euclidean. In consequence, an axiomatic system is changed what is precisely presented in our paper [1]. In this manner, that approach can be applied not only to express knowledge, but can also be nicely adapted to reason about beliefs.

4 Conclusions

In the paper we introduce the original model of belief gradation which we aim to use in formalization of a *persuasion process*. Moreover, we give a succinct presentation of selected formalisms used to express the shades of uncertainty. We analyze multi-valued, probabilistic and graded modalities' approaches to check which of them fulfill the requirements specified for our model.

The first *requirement* says that degrees of beliefs must be included directly in the syntax of a logic since in a specification of formulas of a persuasion process we must indicate changes of the grades of agents' beliefs. The next requirement postulates a need of nesting belief modalities with different degrees, because it enables to describe a subjective success in convincing. The third condition deals with the problem of determining the grades of beliefs in semantics. In our model we assume that a degree expresses a ratio of a number of states accessible via belief relation and verifying a thesis to a number of all states accessible via this relation. The fourth requirement focuses on the fact that a persuasion may change not only the possible visions of the reality, but also belief in their likelihood. The last postulate expresses that the logic should have complete axiomatization. It is essential when we plan to employ a formalism not only for specification but also for formal verification of properties of a persuasion process.

The research presented in the paper enables to understand what formalization of graded beliefs should be used to describe the persuasion in the appropriate manner. The closest in satisfying the preferred requirements is perhaps the Hoek-Meyer proposal. Despite a weakness with respect to expressing the ratio, its strong advantage is the well-known formulation of axioms. On the contrary, the axiomatization of the probabilistic logic is not intuitive. As a result, the attributes of agent's beliefs are not studied profoundly. In this moment of our studies, we think probabilistic and graded modalities' logics best suit our needs (although none of them is a perfect tool). In turn, the multi-valued approach is not adequate for a theory of persuasion since it fails to satisfy the requirement of expressing the grades of beliefs in a language. In the future work, we plan to design implementation for studying the process of convincing with application of different possible formalizations of beliefs' gradation. This will provide a next level for our research by allowing us to *experimentally* compare those proposals.

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